# A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem

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Andrew Beal is a Dallas banker who has a general interest in mathematics and its status within our culture. He also has a personal interest in the discipline. In fact, he has formulated a conjecture in number theory on which he has been working for several years. It is remarkable that occasionally someone working in isolation and with no connections to the mathematical world formulates a problem so close to current research activity.

## The Beal Conjecture

*Let $A, B, C, x, y,$ and $z$ be positive integers with $x, y, z > 2$. If $A^x + B^y = C^z$, then A, B, and C have a common factor.*

Or, slightly restated:

*The equation $A^x + B^y = C^z$ has no solution in positive integers $A, B, C, x, y,$ and $z$ with $x, y,$ and $z$ at least 3 and $A, B,$ and $C$ coprime.*

It turns out that very similar conjectures have been made over the years. In fact, Brun in his 1914 paper states several similar problems [1]. However, it is very timely that this problem be raised now, since Fermat's Last Theorem has just recently been proved (or re-proved) by Wiles [6]. Some of the significant advances made on some problems closely related to the prize problem by Darmon and Granville [2] are indicated below. Darmon and Granville in their article also discuss some related conjectures along this line and provide many relevant references.

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**The prize**. Andrew Beal is very generously offering a prize of \$5,000 for the solution of this problem. The value of the prize will increase by \$5,000 per year up to \$50,000 until it is solved. The prize committee consists of Charles Fefferman, Ron Graham, and R. Daniel Mauldin, who will act as the chair of the committee. All proposed solutions and inquiries about the prize should be sent to Mauldin.

**The abc conjecture**. During the 1980s a conjectured diophantine inequality, the "abc conjecture", with many applications was formulated by Masser, Oesterle, and Szpiro. A survey of this idea has been given by Lang [5] and an elementary discussion by Goldfeld [4]. This inequality can be stated in very simple terms, and it can be applied to Beal's problem. To state the abc conjecture, let us say that if *a*, *b*, and *c* are positive integers, then *N(a,b,c)* denotes the square free part of the product $abc$. In other words, *N(a,b,c)* is the product of the prime divisors of *a*, *b*, and *c* with each divisor counted only once. The abc conjecture can be formulated as follows:

*For each $\epsilon > 0$, there is a constant $\mu > 1$ such that if a and b are relatively prime (or coprime) and c = a+b, then*

$$\max(|a|, |b|, |c|) \leq \mu N(a, b, c)^{1+\epsilon}.$$

Now let us show that *if the abc conjecture holds, then there are no solutions to the prize problem when the exponents are large enough.*

Let $k = \log\mu/\log 2 + (3 + 3\epsilon)$. Let $\min(x, y, z) > k$. Assume A, B, and C are positive integers with A and B relatively prime and such that $A^x + B^y = C^z$. Setting $a = A^x$ and $b = B^y$, we have

$c = a + b = C^z$. From the abc conjecture and the fact that $N(A^x, B^y, C^z) \leq ABC$, we have

$$\max(A^x, B^y, C^z) \leq \mu(ABC)^{1+\epsilon}.$$

If $\max(A, B, C) = A$, then we would have

$$A^x \leq \mu A^{3+3\epsilon}$$

or

$$x \leq \frac{\log \mu}{\log A} + 3 + 3\epsilon \leq k,$$

which is not the case. A similar argument for the other two possibilities for the maximum shows that our original assumption is impossible.

Next let us give an explicit version of the abc conjecture: If $a$ and $b$ are coprime positive integers and $c = a+b$, then $c \leq (N(a, b, c))^2$. Let us see what this implies for the prize problem. Suppose $A^x + B^y = C^z$, with $x \leq y \leq z$. Again, since $A^x$ and $B^y$ are coprime,

$$C^z \leq (N(A^xB^yC^z))^2 \leq (ABC)^2 < C^{2(z/x+z/y+1)}.$$

So $1/2 < 1/x + 1/y + 1/z$. Since $x, y$, and $z$ are greater than 2, we have the following possibilities for (x,y,z): $(3, 3, z > 3), (3, 4, z \geq 4), (3, 5, z \geq 5)$, $(3, 6, z \geq 7), (4, 4, z \geq 5)$, and a finite list of other cases.

**There are only finitely many possible solutions**. In 1995 Darmon and Granville [2] showed that if the positive integers $x, y$, and $z$ are such that $1/x + 1/y + 1/z < 1$, then there are only finitely many triples of coprime integers $A, B, C$ satisfying $A^x + B^y = C^z$. Since each of $x, y$, and $z$ is greater than 2, then $1/x + 1/y + 1/z < 1$ unless $x = y = z = 3$. But Euler and possibly Fermat knew there are no solutions in this case. So for each triple $x, y$, and $z$, all greater than 2, there can be only finitely many solutions to the diophantine equation $A^x + B^y = C^z$.

**Related problems**. What happens if it is only required that $x, y$, and $z$ be $\geq 2$ and at least one of them is greater than 2 and $A$, $B$, and $C$ are coprime? There is a detailed analysis in [2] of those cases where $x, y, z \geq 2$ and $1/x + 1/y + 1/z > 1$.

What happens if we require only that $1/x + 1/y + 1/z < 1$ and $A, B$, and $C$ are coprime? This problem is also discussed by Darmon and Granville. In fact, they have formulated

***The Fermat-Catalan Conjecture.*** There are only finitely many triples of coprime integer powers $x^p, y^q, z^r$ for which

$$x^p + y^q = z^r \text{ with } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

So far, as mentioned in [2], ten solutions have been found. The first five are small solutions. They are $1 + 2^3 = 3^2, 2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2$.

Also five large solutions have been found: $17^7 + 76271^3 = 21063928^2$, $1414^3 + 2213459^2 = 65^7$, $9262^3 + 15312283^2 = 113^7$, $43^8 + 96222^3 = 30042907^2$, $33^8 + 1549034^2 = 15613^3$. The last five big solutions were found by Beukers and Zagier.

Recently Darmon and Merel have shown that there are no coprime solutions with exponents $(x, x, 3)$ with $x \geq 3$ [3].

**Acknowledgment**. Since I am not an expert in this field, I would like to thank Andrew Granville and Richard Guy for their expert help in preparing this note.

## References

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[3] H. Darmon and L. Merel, *Winding quotients and some variants of Fermat's Last Theorem*, preprint.

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[5] S. Lang, *Old and new conjectured diophantine inequalities*, Bull. Amer. Math. Soc. **23** (1990), 37–75.

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*Andrew Beal is a number theory enthusiast residing in Dallas, Texas. He grew up in Lansing, Michigan, and attended Michigan State University. He has a particular interest in some of Fermat's work and has spent many, many hours thinking about Fermat's Last Theorem. He believes that Fermat did possess a relatively simple non-geometry-based proof for FLT, and he continues to search for it. He also believes that Fermat had a method of solution for Pell's equation that remains unknown and that was a function of the squares whose sum equals the coefficient.*

*Andrew is forty-four years old. He and his wife, Simona, have five children. He is the founder/chairman/owner of Beal Bank, Dallas's largest locally owned bank. He is also the recent founder/CEO/owner of Beal Aerospace, which is designing and building a next-generation rocket for launching satellites into earth orbits.*

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